

ON THE CONSTANTS OF THE BOHNENBLUST-HILLE AND HARDY-LITTLEWOOD INEQUALITIES

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ABSTRACT. In this paper, among other results, we improve the best known estimates for the constants of the generalized Bohnenblust-Hille inequality. These enhancements are then used to improve the best known constants of the Hardy-Littlewood inequality; this inequality asserts that for a positive integer $m \geq 2$ with $2m \leq p \leq \infty$ and $\mathbb{K} = \mathbb{R}$ or \mathbb{C} there exists a constant $C_{m,p}^{\mathbb{K}} \geq 1$ such that, for all continuous m -linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \rightarrow \mathbb{K}$, and all positive integers n ,

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p}^{\mathbb{K}} \|T\|,$$

and the exponent $\frac{2mp}{mp+p-2m}$ is sharp. In particular, we show that for $p > 2m^3 - 4m^2 + 2m$ the optimal constants satisfying the above inequality are dominated by the best known estimates for the constants of the m -linear Bohnenblust-Hille inequality. More precisely if γ denotes the Euler-Mascheroni constant, considering the case of complex scalars as an illustration, we show that

$$C_{m,p}^{\mathbb{C}} \leq \prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}} < m^{\frac{1-\gamma}{2}},$$

which is somewhat surprising since this new formula has no dependence on p (the former estimate depends on p but, paradoxally, is worse than this new one). This suggest the following open problems:

- 1) Are the optimal constants of the Hardy-Littlewood inequality and Bohnenblust-Hille inequalities the same?
- 2) Are the optimal constants of the Hardy-Littlewood inequality independent of p (at least for large p)?

1. INTRODUCTION

Let \mathbb{K} be \mathbb{R} or \mathbb{C} and $m \geq 2$ be a positive integer. In 1931, F. Bohnenblust and E. Hille (see [7]) proved in the *Annals of Mathematics* that there exists a constant $B_{\mathbb{K},m}^{\text{mult}} \geq 1$ such that for all continuous m -linear forms $T : \ell_{\infty}^n \times \cdots \times \ell_{\infty}^n \rightarrow \mathbb{K}$, and all positive integers n ,

$$(1) \quad \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{mult}} \|T\|.$$

The task of estimating the constants $B_{\mathbb{K},m}^{\text{mult}}$ of this inequality (now known as the Bohnenblust-Hille inequality) is nowadays a challenging problem in Mathematical Analysis. For complex scalars, having good estimates for the polynomial version of $B_{\mathbb{K},m}^{\text{mult}}$ is crucial in applications in Complex Analysis and Analytic Number Theory (see [10]); for real scalars, the estimates of $B_{\mathbb{R},m}^{\text{mult}}$ are important in Quantum Information Theory (see [19]). In the last years a series of papers related to the Bohnenblust-Hille inequality have been published

The authors are supported by CNPq Grant 313797/2013-7 - PVE - Linha 2.

and several advances were achieved (see [1, 9, 10, 11, 21, 22, 23] and the references therein). For instance, the subexponentiality of the constants of the polynomial version of the Bohnenblust–Hille inequality (case of complex scalars) was recently used in [5] to obtain the asymptotic growth of the Bohr radius of the n -dimensional polydisk. More precisely, according to Boas and Khavinson [6], the Bohr radius K_n of the n -dimensional polydisk is the largest positive number r such that all polynomials $\sum_{\alpha} a_{\alpha} z^{\alpha}$ on \mathbb{C}^n satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_{\alpha} |a_{\alpha} z^{\alpha}| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_{\alpha} a_{\alpha} z^{\alpha} \right|.$$

The Bohr radius K_1 was estimated by H. Bohr, and it was later shown independently by M. Riesz, I. Schur and F. Wiener that $K_1 = 1/3$ (see [6, 8] and the references therein). For $n \geq 2$, exact values of K_n are unknown. In [5], the subexponentiality of the constants of the complex polynomial version of the Bohnenblust–Hille inequality was proved and using this fact it was finally proved that

$$\lim_{n \rightarrow \infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1,$$

solving a challenging problem that many researchers have been chipping away at for several years.

The best known estimates for the constants in (1), which are recently presented in [5], are

$$\begin{aligned} B_{\mathbb{C},m}^{\text{mult}} &\leq \prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ (2) \quad B_{\mathbb{R},m}^{\text{mult}} &\leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}}, \quad \text{for } m \geq 14, \\ B_{\mathbb{R},m}^{\text{mult}} &\leq \prod_{j=2}^m 2^{\frac{1}{2j-2}}, \quad \text{for } 2 \leq m \leq 13. \end{aligned}$$

In a more friendly presentation the above formulas tell us that the growth of the constants $B_{\mathbb{K},m}^{\text{mult}}$ is subpolynomial (in fact, sublinear) since, from the above estimates it can be proved that (see [5])

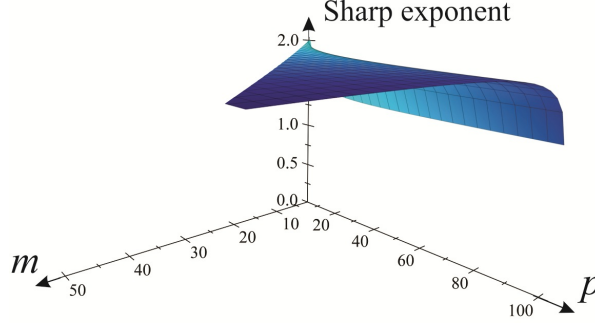
$$\begin{aligned} B_{\mathbb{C},m}^{\text{mult}} &< m^{\frac{1-\gamma}{2}} < m^{0.21139}, \\ B_{\mathbb{R},m}^{\text{mult}} &< 1.3 \cdot m^{\frac{2-\log 2-\gamma}{2}} < 1.3 \cdot m^{0.36482}, \end{aligned}$$

where γ denotes the Euler–Mascheroni constant. This is a quite surprising result since the previous estimates (from 1931–2011) predicted an exponential growth; it was only in 2012, with [22], motivated by [11], that the panorama started to change.

The Hardy–Littlewood inequality is a natural extension of the Bohnenblust–Hille inequality (see [1, 16, 24]) and asserts that for $4 \leq 2m \leq p \leq \infty$ there exists a constant $C_{m,p}^{\mathbb{K}} \geq 1$ such that, for all continuous m -linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \rightarrow \mathbb{K}$, and all positive integers n ,

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C_{m,p}^{\mathbb{K}} \|T\|,$$

and the exponent $\frac{2mp}{mp+p-2m}$ is optimal.



The original estimates for $C_{m,p}^{\mathbb{K}}$ were of the form

$$(3) \quad C_{m,p}^{\mathbb{K}} \leq (\sqrt{2})^{m-1}$$

(see [1]); nowadays the best known estimates for the constants $C_{m,p}^{\mathbb{K}}$ (see [4]) are

$$(4) \quad \begin{aligned} C_{m,p}^{\mathbb{C}} &\leq \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}\right)^{\frac{p-2m}{p}}, \\ C_{m,p}^{\mathbb{R}} &\leq (\sqrt{2})^{\frac{2m(m-1)}{p}} \left(2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}\right)^{\frac{p-2m}{p}}, \quad \text{for } m \geq 14, \\ C_{m,p}^{\mathbb{R}} &\leq (\sqrt{2})^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^m 2^{\frac{1}{2j-2}}\right)^{\frac{p-2m}{p}}, \quad \text{for } 2 \leq m \leq 13. \end{aligned}$$

Note that the presence of the parameter p in the formulas of (4), if compared to (3), catches more subtle information since now it is clear that the estimates become “better” as p grows. As p tends to infinity we note that the above estimates tend to best known estimates for $B_{\mathbb{K},m}^{\text{mult}}$ (see (2)). In this paper, among other results, we show that for $p > 2m^3 - 4m^2 + 2m$ the constant $C_{m,p}^{\mathbb{K}}$ has the exactly same upper bounds that we have for the Bohnenblust–Hille constants (2). More precisely we shall show that if $p > 2m^3 - 4m^2 + 2m$, then

$$(5) \quad \begin{aligned} C_{m,p}^{\mathbb{C}} &\leq \prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ C_{m,p}^{\mathbb{R}} &\leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}, \quad \text{for } m \geq 14, \\ C_{m,p}^{\mathbb{R}} &\leq \prod_{j=2}^m 2^{\frac{1}{2j-2}}, \quad \text{for } 2 \leq m \leq 13. \end{aligned}$$

It is not difficult to verify that (5) in fact improves (4). However the most interesting point is that in (5), contrary to (4), we have no dependence on p in the formulas and, besides, these new estimates are precisely the best known estimates for the constants of the Bohnenblust–Hille inequality (see (2)).

To prove these new estimates we also improve the best known estimates for the generalized Bohnenblust–Hille inequality. Recall that the generalized Bohnenblust–Hille inequality (see [1]) asserts that if $(q_1, \dots, q_m) \in [1, 2]^m$ are so that

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{m+1}{2},$$

then there is $B_{m,(q_1,\dots,q_m)}^{\mathbb{K}} \geq 1$ such that

$$\left(\sum_{j_1=1}^n \dots \left(\sum_{j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{q_m} \right)^{\frac{q_m-1}{q_m}} \dots \right)^{\frac{1}{q_1}} \leq B_{m,(q_1,\dots,q_m)}^{\mathbb{K}} \|T\|$$

for all m -linear forms $T : \ell_{\infty}^n \times \dots \times \ell_{\infty}^n \rightarrow \mathbb{K}$, and all positive integers n . The importance of this result transcends the intrinsic mathematical novelty since, as it was recently shown (see [5]), this new approach is fundamental to improve the estimates of the constants of the classical Bohnenblust–Hille inequality. The best known estimates for the constants $B_{m,(q_1,\dots,q_m)}^{\mathbb{K}}$ are presented in [2]. More precisely, for complex scalars and $1 \leq q_1 \leq \dots \leq q_m \leq 2$, from [2] we know that, for $\mathbf{q} = (q_1, \dots, q_m)$,

$$B_{m,\mathbf{q}}^{\mathbb{C}} \leq \left(\prod_{j=1}^m \Gamma \left(2 - \frac{1}{j} \right)^{\frac{j}{2-2j}} \right)^{2m \left(\frac{1}{q_m} - \frac{1}{2} \right)} \left(\prod_{k=1}^{m-1} \left(\Gamma \left(\frac{3k+1}{2k+2} \right)^{\left(\frac{-k-1}{2k} \right)(m-k)} \prod_{j=1}^k \Gamma \left(2 - \frac{1}{j} \right)^{\frac{j}{2-2j}} \right)^{2k \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right)} \right).$$

In the present paper we improve the above estimates for a certain family of (q_1, \dots, q_m) . More precisely, if

$$\max q_i < \frac{2m^2 - 4m + 2}{m^2 - m - 1},$$

then

$$B_{m,(q_1,\dots,q_m)}^{\mathbb{C}} \leq \prod_{j=2}^m \Gamma \left(2 - \frac{1}{j} \right)^{\frac{j}{2-2j}}.$$

A similar result holds for real scalars. These results have a crucial importance in the next sections.

The paper is organized as follows. In Section 2 we obtain new estimates for the generalized Bohnenblust–Hille inequality and in Section 3 we use these estimates to prove new estimates for the constants of the Hardy–Littlewood inequality. In the final section (Section 4) the estimates of the previous sections are used to obtain new constants for the generalized Hardy–Littlewood inequality.

2. NEW ESTIMATES FOR THE CONSTANTS OF THE GENERALIZED BOHNENBLUST-HILLE INEQUALITY

We recall that the Khinchine inequality (see [12]) asserts that for any $0 < q < \infty$, there are positive constants A_q, B_q such that regardless of the scalar sequence $(a_j)_{j=1}^n$ we have

$$A_q \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{j=1}^n a_j r_j(t) \right|^q dt \right)^{\frac{1}{q}} \leq B_q \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}},$$

where r_j are the Rademacher functions. More generally, from the above inequality together with the Minkowski inequality we know that (see [4], for instance, and the references therein)

$$(6) \quad A_q^m \left(\sum_{j_1, \dots, j_m=1}^n |a_{j_1 \dots j_m}|^2 \right)^{\frac{1}{2}} \leq \left(\int_I \left| \sum_{j_1, \dots, j_m=1}^n a_{j_1 \dots j_m} r_{j_1}(t_1) \dots r_{j_m}(t_m) \right|^q dt \right)^{\frac{1}{q}} \leq B_q^m \left(\sum_{j_1, \dots, j_m=1}^n |a_{j_1 \dots j_m}|^2 \right)^{\frac{1}{2}},$$

where $I = [0, 1]^m$ and $dt = dt_1 \dots dt_m$, for all scalar sequences $(a_{j_1 \dots j_m})_{j_1, \dots, j_m=1}^n$.

The best constants A_q are known (see [15]). Indeed,

- $A_q = \sqrt{2} \left(\frac{\Gamma(\frac{1+q}{2})}{\sqrt{\pi}} \right)^{\frac{1}{q}}$ if $q > q_0 \cong 1.8474$;
- $A_q = 2^{\frac{1}{2} - \frac{1}{q}}$ if $q < q_0$.

The definition of the number q_0 above is the following: $q_0 \in (1, 2)$ is the unique real number with

$$\Gamma\left(\frac{p_0 + 1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

For complex scalars, using Steinhaus variables instead of Rademacher functions it is well known that a similar inequality holds, but with better constants (see [17, 25]). In this case the optimal constant is

- $A_q = \Gamma\left(\frac{q+2}{2}\right)^{\frac{1}{q}}$ if $q \in [1, 2]$.

The notation of the constant A_q above will be used in all this paper.

Lemma 2.1. *Let $(q_1, \dots, q_m) \in [1, 2]^m$ such that $\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{m+1}{2}$. If $q_i \geq \frac{2m-2}{m}$ for some index i and $q_k = q_l$ for all $k \neq i$ and $l \neq i$, then*

$$B_{m, (q_1, \dots, q_m)}^{\mathbb{K}} \leq \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1},$$

where $A_{\frac{2j-2}{j}}$ are the respective constants of the Khinchine inequality.

Proof. There is no loss of generality in supposing that $i = 1$. By using the multiple Khinchine inequality (6) we have (see [4, Section 2] for details)

$$\left(\sum_{j_1, \dots, j_{m-1}=1}^n \left(\sum_{j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^2 \right)^{\frac{1}{2} \frac{2m-2}{m}} \right)^{\frac{m}{2m-2}} \leq A_{\frac{2m-2}{m}}^{-1} B_{\mathbb{K}, m-1}^{\text{mult}} \|T\|.$$

From [5] we know that

$$B_{\mathbb{K}, 1}^{\text{mult}} = 1 \quad \text{and} \quad B_{\mathbb{K}, m}^{\text{mult}} \leq A_{\frac{2m-2}{m}}^{-1} B_{\mathbb{K}, m-1}^{\text{mult}},$$

and thus

$$A_{\frac{2m-2}{m}}^{-1} B_{\mathbb{K}, m-1}^{\text{mult}} = \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}.$$

Taking the m exponents

$$\begin{aligned} & \left(\frac{2m-2}{m}, \dots, \frac{2m-2}{m}, 2 \right), \\ & \quad \vdots \\ & \left(2, \frac{2m-2}{m}, \dots, \frac{2m-2}{m} \right), \end{aligned}$$

interpolated (in the sense of [1]) with $\theta_1 = \dots = \theta_{m-1} = \frac{2}{q_1} - 1$ and $\theta_m = m - \frac{2m-2}{q_1}$, we conclude that the exponent obtained is (q_1, \dots, q_m) . Since $\frac{2m-2}{m} < 2$, from a repeated use of the Minkowski inequality (in the lines of the arguments from [1]) we know that the constants associated to all the above exponents are dominated by $\prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}$, and the proof is done. \square

From now on, for any function f , whenever it makes sense we formally define $f(\infty) = \lim_{p \rightarrow \infty} f(p)$.

Lemma 2.2. *Let $m \geq 2$ be a positive integer, let $2m < p \leq \infty$, let $q_1, \dots, q_m \in \left[\frac{p}{p-m}, 2 \right]$. If*

$$(7) \quad \frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{mp + p - 2m}{2p},$$

then, for all $s \in (\max q_i, 2]$, the vector $(q_1^{-1}, \dots, q_m^{-1})$ belongs to the convex hull in \mathbb{R}^m of

$$\left\{ \sum_{k=1}^m a_{1k} e_k, \dots, \sum_{k=1}^m a_{mk} e_k \right\},$$

where

$$a_{jk} = \begin{cases} s^{-1}, & \text{if } k \neq j \\ \lambda_{m,s}^{-1}, & \text{if } k = j \end{cases}$$

and

$$\lambda_{m,s} = \frac{2ps}{mps + ps + 2p - 2mp - 2ms}.$$

Equivalently, we say that the exponent (q_1, \dots, q_m) is the interpolation of the m exponents $(s, \dots, s, \lambda_{m,s}), \dots, (\lambda_{m,s}, s, \dots, s)$.

Proof. We want to prove that for $(q_1, \dots, q_m) \in \left[\frac{p}{p-m}, 2\right]^m$ and $s \in (\max q_i, 2]$ there are $0 < \theta_{j,s} < 1$, $j = 1, \dots, m$, such that

$$\begin{aligned} \sum_{j=1}^m \theta_{j,s} &= 1, \\ \frac{1}{q_1} &= \frac{\theta_{1,s}}{\lambda_{m,s}} + \frac{\theta_{2,s}}{s} + \dots + \frac{\theta_{m,s}}{s}, \\ &\vdots \\ \frac{1}{q_m} &= \frac{\theta_{1,s}}{s} + \dots + \frac{\theta_{m-1,s}}{s} + \frac{\theta_{m,s}}{\lambda_{m,s}}. \end{aligned}$$

Observe initially that from (7) we have

$$\max q_i \geq \frac{2mp}{mp+p-2m}.$$

Note also that for all $s \in \left[\frac{2mp-2p}{mp-2m}, 2\right]$ we have

$$(8) \quad mps + ps + 2p - 2mp - 2ms > 0 \quad \text{and} \quad \frac{p}{p-m} \leq \lambda_{m,s} \leq 2.$$

Since $s > \max q_i \geq \frac{2mp}{mp+p-2m} > \frac{2mp-2p}{mp-2m}$ (the last inequality is strict because we are not considering the case $p = 2m$) it follows that $\lambda_{m,s}$ is well defined for all $s \in (\max q_i, 2]$. Furthermore, for all $s > \frac{2mp}{mp+p-2m}$ it is possible to prove that $\lambda_{m,s} < s$. In fact, $s > \frac{2mp}{mp+p-2m}$ implies $mps + ps - 2ms > 2mp$ and thus adding $2p$ in both sides of this inequality we can conclude that

$$\frac{2ps}{mps + ps + 2p - 2mp - 2ms} < \frac{2ps}{2p} = s,$$

i.e.,

$$(9) \quad \lambda_{m,s} < s.$$

For each $j = 1, \dots, m$, consider

$$\theta_{j,s} = \frac{\lambda_{m,s}(s - q_j)}{q_j(s - \lambda_{m,s})}.$$

Since $\sum_{j=1}^m \frac{1}{q_j} = \frac{mp+p-2m}{2p}$ we conclude that

$$\begin{aligned} \sum_{j=1}^m \theta_{j,s} &= \sum_{j=1}^m \frac{\lambda_{m,s}(s - q_j)}{q_j(s - \lambda_{m,s})} \\ &= \frac{\lambda_{m,s}}{s - \lambda_{m,s}} \left(s \sum_{j=1}^m \frac{1}{q_j} - m \right) \\ &= 1. \end{aligned}$$

Since by hypothesis $s > \max q_i \geq q_j$ for all $j = 1, \dots, m$, it follows that $\theta_{j,s} > 0$ for all $j = 1, \dots, m$ and thus

$$0 < \theta_{j,s} < \sum_{j=1}^m \theta_{j,s} = 1.$$

Finally, note that

$$\frac{\theta_{j,s}}{\lambda_{m,s}} + \frac{1 - \theta_{j,s}}{s} = \frac{\frac{\lambda_{m,s}(s-q_j)}{q_j(s-\lambda_{m,s})}}{\lambda_{m,s}} + \frac{1 - \frac{\lambda_{m,s}(s-q_j)}{q_j(s-\lambda_{m,s})}}{s} = \frac{1}{q_j}.$$

Therefore

$$\begin{aligned} \frac{1}{q_1} &= \frac{\theta_{1,s}}{\lambda_{m,s}} + \frac{\theta_{2,s}}{s} + \dots + \frac{\theta_{m,s}}{s}, \\ &\vdots \\ \frac{1}{q_m} &= \frac{\theta_{1,s}}{s} + \dots + \frac{\theta_{m-1,s}}{s} + \frac{\theta_{m,s}}{\lambda_{m,s}}, \end{aligned}$$

and the proof is done. □

Combining the two previous lemmata we have:

Theorem 2.3. *Let $m \geq 2$ be a positive integer and $q_1, \dots, q_m \in [1, 2]$. If*

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{m+1}{2},$$

and

$$\max q_i < \frac{2m^2 - 4m + 2}{m^2 - m - 1},$$

then

$$B_{m,(q_1, \dots, q_m)}^{\mathbb{K}} \leq \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1},$$

where $A_{\frac{2j-2}{j}}$ are the respective constants of the Khnichine inequality.

Proof. Let

$$s = \frac{2m^2 - 4m + 2}{m^2 - m - 1} \quad \text{and} \quad q = \frac{2m - 2}{m}.$$

Since

$$\frac{m-1}{s} + \frac{1}{q} = \frac{m+1}{2},$$

from Lemma 2.1 the Bohnenblust–Hille exponents

$$(t_1, \dots, t_m) = (s, \dots, s, q), \dots, (q, s, \dots, s)$$

are associated to

$$B_{m,(t_1, \dots, t_m)}^{\mathbb{K}} \leq \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}.$$

Since by hypothesis

$$\max q_i < \frac{2m^2 - 4m + 2}{m^2 - m - 1} = s,$$

from the previous lemma (Lemma 2.2) with $p = \infty$, the exponent (q_1, \dots, q_m) is the interpolation of

$$\left(\frac{2s}{ms + s + 2 - 2m}, s, \dots, s \right), \dots, \left(s, \dots, s, \frac{2s}{ms + s + 2 - 2m} \right).$$

But note that

$$\frac{2s}{ms + s + 2 - 2m} = \frac{2m - 2}{m}$$

and from Lemma 2.1 they are associated to the constants

$$B_{m,(q_1,\dots,q_m)}^{\mathbb{K}} \leq \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}.$$

□

Corollary 2.4. *Let $m \geq 2$ be a positive integer and $q_1, \dots, q_m \in [1, 2]$. If*

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{m+1}{2},$$

and

$$\max q_i < \frac{2m^2 - 4m + 2}{m^2 - m - 1},$$

then

$$\begin{aligned} B_{m,(q_1,\dots,q_m)}^{\mathbb{C}} &\leq \prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ B_{m,(q_1,\dots,q_m)}^{\mathbb{R}} &\leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}, \quad \text{for } m \geq 14, \\ B_{m,(q_1,\dots,q_m)}^{\mathbb{R}} &\leq \prod_{j=2}^m 2^{\frac{1}{2j-2}}, \quad \text{for } 2 \leq m \leq 13. \end{aligned}$$

3. APPLICATION 1: IMPROVING THE CONSTANTS OF THE HARDY-LITTLEWOOD INEQUALITY

The main result of this section shows that for $p > 2m^3 - 4m^2 + 2m$ the optimal constants satisfying the Hardy–Littlewood inequality for m -linear forms in ℓ_p spaces are dominated by the best known estimates for the constants of the m -linear Bohnenblust–Hille inequality; this result improves the recent estimates (see (4)), and may suggest a more subtle connection between the optimal constants of these inequalities.

Theorem 3.1. *Let $m \geq 2$ be a positive integer and $2m^3 - 4m^2 + 2m < p \leq \infty$. Then, for all continuous m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n , we have*

$$(10) \quad \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{m+p-2m}} \right)^{\frac{m+p-2m}{2mp}} \leq \left(\prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1} \right) \|T\|.$$

Proof. The case $p = \infty$ in (10) is precisely the Bohnenblust–Hille inequality, so we just need to consider $2m^3 - 4m^2 + 2m < p < \infty$. Let $\frac{2m-2}{m} \leq s \leq 2$ and

$$\lambda_{0,s} = \frac{2s}{ms + s + 2 - 2m}.$$

Note that

$$(11) \quad ms + s + 2 - 2m > 0 \quad \text{and} \quad 1 \leq \lambda_{0,s} \leq 2.$$

Since

$$\frac{m-1}{s} + \frac{1}{\lambda_{0,s}} = \frac{m+1}{2},$$

from the generalized Bohnenblust–Hille inequality (see [1]) we know that there is a constant $C_m \geq 1$ such that for all m -linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \rightarrow \mathbb{K}$ we have, for all $i = 1, \dots, m$,

$$(12) \quad \left(\sum_{j_i=1}^n \left(\sum_{\hat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{0,s}} \right)^{\frac{1}{\lambda_{0,s}}} \leq C_m \|T\|.$$

Above, $\sum_{\hat{j}_i=1}^n$ means the sum over all j_k for all $k \neq i$. If we choose $s = \frac{2mp}{mp+p-2m}$ (note that this s belongs to the interval $[\frac{2m-2}{m}, 2]$), we have $s > \frac{2m}{m+1}$ (this inequality is strict because we are considering the case $p < \infty$) and thus $\lambda_{0,s} < s$. In fact, $s > \frac{2m}{m+1}$ implies $ms + s > 2m$ and thus adding 2 in both sides of this inequality we can conclude that

$$\frac{2s}{ms + s + 2 - 2m} < \frac{2s}{2} = s,$$

i.e.,

$$(13) \quad \lambda_{0,s} < s.$$

Since $p > 2m^3 - 4m^2 + 2m$ we conclude that

$$s < \frac{2m^2 - 4m + 2}{m^2 - m - 1}.$$

Thus, from Theorem 2.3, the optimal constant associated to the multiple exponent

$$(\lambda_{0,s}, s, s, \dots, s)$$

is less than or equal to

$$C_m = \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}.$$

More precisely, (12) is valid with C_m as above. Now the proof follows the same lines, *mutatis mutandis*, of the proof of [4, Theorem 1.1]. \square

Remark 3.2. Note that it is simple to verify that these new estimates are better than the old estimates. In fact, for complex scalars the inequality

$$\prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1} < \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1} \right)^{\frac{p-2m}{p}}$$

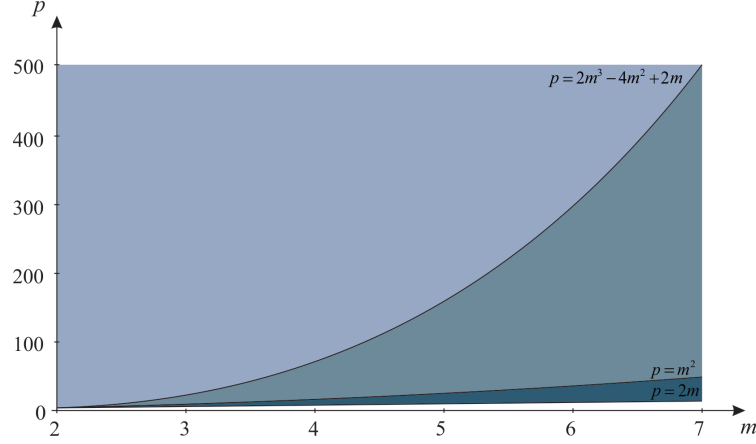
is a straightforward consequence of

$$\prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1} < \left(\frac{2}{\sqrt{\pi}} \right)^{m-1},$$

which is true for $m \geq 3$. The case of real scalars is analogous.

Recall that from [4] we know that for $p \geq m^2$ the constants of the Hardy–Littlewood inequality have a subpolynomial growth. The following graph illustrates what we have thus far, combined with Theorem 3.1.

A question that arises naturally is: Are the optimal constants of the Hardy–Littlewood and Bohnenblust–Hille inequalities the same? This result is maybe slightly suggested by the above estimates. In addition,



$$\begin{aligned}
 & \text{■ } C_{m,p}^{\mathbb{C}} \leq \prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}} \quad (\text{subpolynomial growth}) \\
 & \text{■ } C_{m,p}^{\mathbb{C}} \leq \left(\frac{2}{\sqrt{p}}\right)^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}\right)^{\frac{p-2m}{p}} \quad (\text{subpolynomial growth}) \\
 & \text{■ } C_{m,p}^{\mathbb{C}} \leq \left(\frac{2}{\sqrt{p}}\right)^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}\right)^{\frac{p-2m}{p}} \quad (\text{exponential growth})
 \end{aligned}$$

the best known lower estimates for the real constants of the Hardy–Littlewood inequality (see [3]) are very similar to the respective lower estimates for the real constants of the Bohnenblust–Hille inequality as it can be seen in [14]. More precisely, from [3, 14] we know that, for $m \geq 2$,

$$C_{m,p}^{\mathbb{R}} > 2^{\frac{mp + (6-4\log_2(1.74))m - 2m^2 - p}{mp}} > 1$$

and

$$B_{\mathbb{R},m}^{\text{mult}} \geq 2^{1-\frac{1}{m}} \geq \sqrt{2}.$$

4. APPLICATION 2: CONSTANTS OF THE GENERALIZED HARDY-LITTLEWOOD INEQUALITY

Given an integer $m \geq 2$, the generalized Hardy–Littlewood inequality (see [1, 16, 24]) asserts that for $2m \leq p \leq \infty$ and $\mathbf{q} := (q_1, \dots, q_m) \in \left[\frac{p}{p-m}, 2\right]^m$ such that

$$(14) \quad \frac{1}{q_1} + \dots + \frac{1}{q_m} \leq \frac{mp + p - 2m}{2p},$$

there exists a constant $C_{m,p,\mathbf{q}}^{\mathbb{K}} \geq 1$ such that, for all continuous m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n ,

$$\left(\sum_{j_1=1}^n \left(\sum_{j_2=1}^n \left(\dots \left(\sum_{j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \dots \right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{m,p,\mathbf{q}}^{\mathbb{K}} \|T\|.$$

The best known estimates for the constants $C_{m,p,\mathbf{q}}^{\mathbb{K}}$ are $(\sqrt{2})^{m-1}$ for real scalars and $\left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ for complex scalars (see [1]). Very recently, in [4] (and in the previous section, see (4)), better constants were obtained when $q_1 = \dots = q_m = \frac{2mp}{mp+p-2m}$. Now we extend the results from [4] to general multiple exponents. Of course the interesting case is the border case, i.e., when we have an equality in (14). The proof is slightly more elaborated than the proof of Theorem 3.1 and also a bit more technical than the proof of the main result of [4].

Theorem 4.1. *Let $m \geq 2$ be a positive integer and $2m < p \leq \infty$. Let also $\mathbf{q} := (q_1, \dots, q_m) \in \left[\frac{p}{p-m}, 2\right]^m$ be such that*

$$\frac{1}{q_1} + \dots + \frac{1}{q_m} = \frac{mp + p - 2m}{2p}.$$

(i) *If $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, then*

$$\begin{aligned} C_{m,p,\mathbf{q}}^{\mathbb{C}} &\leq \prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ C_{m,p,\mathbf{q}}^{\mathbb{R}} &\leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}, \quad \text{if } m \geq 14, \\ C_{m,p,\mathbf{q}}^{\mathbb{R}} &\leq \prod_{j=2}^m 2^{\frac{1}{2j-2}}, \quad \text{if } 2 \leq m \leq 13. \end{aligned}$$

(ii) *If $\max q_i \geq \frac{2m^2-4m+2}{m^2-m-1}$, then*

$$\begin{aligned} C_{m,p,\mathbf{q}}^{\mathbb{C}} &\leq \left(\frac{2}{\sqrt{\pi}}\right)^{2(m-1)\left(\frac{m+1}{2} - \frac{m}{\max q_i}\right)} \left(\prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}\right)^{m\left(\frac{2}{\max q_i} - 1\right)}, \\ C_{m,p,\mathbf{q}}^{\mathbb{R}} &\leq 2^{(m-1)\left(\frac{m+1}{2} - \frac{m}{\max q_i}\right)} \left(2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}\right)^{m\left(\frac{2}{\max q_i} - 1\right)}, \quad \text{if } m \geq 14, \\ C_{m,p,\mathbf{q}}^{\mathbb{R}} &\leq 2^{(m-1)\left(\frac{m+1}{2} - \frac{m}{\max q_i}\right)} \left(\prod_{j=2}^m 2^{\frac{1}{2j-2}}\right)^{m\left(\frac{2}{\max q_i} - 1\right)}, \quad \text{if } 2 \leq m \leq 13. \end{aligned}$$

Proof. Let us first suppose $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$. The arguments follow the general lines of [4], but are slightly different and due the technicalities we present the details for the sake of clarity. Define for $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$,

$$(15) \quad \lambda_{m,s} = \frac{2ps}{mps + ps + 2p - 2mp - 2ms}.$$

Observe that $\lambda_{m,s}$ is well defined for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$. In fact, as we have in (8) note that for all $s \in \left[\frac{2mp-2p}{mp-2m}, 2\right]$ we have

$$mps + ps + 2p - 2mp - 2ms > 0 \quad \text{and} \quad \frac{p}{p-m} \leq \lambda_{m,s} \leq 2.$$

Since $s > \max q_i \geq \frac{2mp}{mp+p-2m} > \frac{2mp-2p}{mp-2m}$ (the last inequality is strict because we are not considering the case $p = 2m$) and $\frac{2m^2-4m+2}{m^2-m-1} \leq 2$ it follows that $\lambda_{m,s}$ is well defined for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$.

Let us prove

$$(16) \quad C_{m,p,(\lambda_{m,s},s,\dots,s)}^{\mathbb{K}} \leq \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}$$

for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$. In fact, for these values of s , consider

$$\lambda_{0,s} = \frac{2s}{ms + s + 2 - 2m}.$$

Observe that if $p = \infty$ then $\lambda_{m,s} = \lambda_{0,s}$. Since

$$\frac{m-1}{s} + \frac{1}{\lambda_{0,s}} = \frac{m+1}{2},$$

from the generalized Bohnenblust–Hille inequality (see [1]) we know that there is a constant $C_m \geq 1$ such that for all m -linear forms $T : \ell_\infty^n \times \dots \times \ell_\infty^n \rightarrow \mathbb{K}$ we have, for all $i = 1, \dots, m$,

$$(17) \quad \left(\sum_{j_i=1}^n \left(\sum_{\hat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{0,s}} \right)^{\frac{1}{\lambda_{0,s}}} \leq C_m \|T\|.$$

Since

$$\frac{2m}{m+1} \leq \frac{2mp}{mp+p-2m} \leq \max q_i < s < \frac{2m^2-4m+2}{m^2-m-1}$$

it is not too difficult to prove that (see (13))

$$\lambda_{0,s} < s < \frac{2m^2-4m+2}{m^2-m-1}.$$

Since $s < \frac{2m^2-4m+2}{m^2-m-1}$ we conclude by Theorem 2.3 that the optimal constant associated to the multiple exponent

$$(\lambda_{0,s}, s, s, \dots, s)$$

is less than or equal to

$$(18) \quad \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}.$$

More precisely, (17) is valid with C_m as above.

Since $\lambda_{m,s} = \lambda_{0,s}$ if $p = \infty$, we have (16) for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$ and the proof is done for this case.

For $2m < p < \infty$, let

$$\lambda_{j,s} = \frac{\lambda_{0,s}p}{p - \lambda_{0,s}j}$$

for all $j = 1, \dots, m$. Note that

$$\lambda_{m,s} = \frac{2ps}{mps + ps + 2p - 2mp - 2ms}$$

and this notation is compatible with (15). Since $s > \max q_i \geq \frac{2mp}{mp+p-2m} \geq \frac{2mp}{mp+p-2j}$ for all $j = 1, \dots, m$ we also observe that

$$(19) \quad \lambda_{j,s} < s$$

for all $j = 1, \dots, m$. Moreover, observe that

$$\left(\frac{p}{\lambda_{j,s}} \right)^* = \frac{\lambda_{j+1,s}}{\lambda_{j,s}}$$

for all $j = 0, \dots, m-1$. Here, as usual, $\left(\frac{p}{\lambda_{j,s}} \right)^*$ denotes the conjugate number of $\left(\frac{p}{\lambda_{j,s}} \right)$. From now on part of the proof of (i) follows the steps of the proof of the main result of [4], but we prefer to show the details for the sake of completeness (note that the final part of the proof of (i) requires a more subtle argument than the one used in [4]).

Let us suppose that $1 \leq k \leq m$ and that

$$\left(\sum_{j_i=1}^n \left(\sum_{\hat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1,s}} \right)^{\frac{1}{\lambda_{k-1,s}}} \leq C_m \|T\|$$

is true for all continuous m -linear forms $T : \underbrace{\ell_p^n \times \dots \times \ell_p^n}_{k-1 \text{ times}} \times \ell_\infty^n \times \dots \times \ell_\infty^n \rightarrow \mathbb{K}$ and for all $i = 1, \dots, m$. Let us prove that

$$\left(\sum_{j_i=1}^n \left(\sum_{\hat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k,s}} \right)^{\frac{1}{\lambda_{k,s}}} \leq C_m \|T\|$$

for all continuous m -linear forms $T : \underbrace{\ell_p^n \times \dots \times \ell_p^n}_{k \text{ times}} \times \ell_\infty^n \times \dots \times \ell_\infty^n \rightarrow \mathbb{K}$ and for all $i = 1, \dots, m$.

The initial case (the case in which all $p = \infty$) is precisely (17) with C_m as in (18).

Consider

$$T \in \mathcal{L}(\underbrace{\ell_p^n, \dots, \ell_p^n}_{k \text{ times}}, \ell_\infty^n, \dots, \ell_\infty^n; \mathbb{R})$$

and for each $x \in B_{\ell_p^n}$ define

$$\begin{aligned} T^{(x)} : \underbrace{\ell_p^n \times \dots \times \ell_p^n}_{k-1 \text{ times}} \times \ell_\infty^n \times \dots \times \ell_\infty^n &\rightarrow \mathbb{R} \\ (z^{(1)}, \dots, z^{(m)}) &\mapsto T(z^{(1)}, \dots, z^{(k-1)}, xz^{(k)}, z^{(k+1)}, \dots, z^{(m)}), \end{aligned}$$

with $xz^{(k)} = (x_j z_j^{(k)})_{j=1}^n$. Observe that

$$\|T\| = \sup\{\|T^{(x)}\| : x \in B_{\ell_p^n}\}.$$

By applying the induction hypothesis to $T^{(x)}$, we obtain

$$\begin{aligned}
 & \left(\sum_{j_i=1}^n \left(\sum_{\hat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s} \lambda_{k-1, s}} \right)^{\frac{1}{\lambda_{k-1, s}}} \\
 &= \left(\sum_{j_i=1}^n \left(\sum_{\hat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_{k-1}}, x e_{j_k}, e_{j_{k+1}}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1, s}} \right)^{\frac{1}{\lambda_{k-1, s}}} \\
 (20) \quad &= \left(\sum_{j_i=1}^n \left(\sum_{\hat{j}_i=1}^n |T^{(x)}(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1, s}} \right)^{\frac{1}{\lambda_{k-1, s}}} \\
 &\leq C_m \|T^{(x)}\| \\
 &\leq C_m \|T\|
 \end{aligned}$$

for all $i = 1, \dots, m$.

We will analyze two cases:

- $i = k$.

Since

$$\left(\frac{p}{\lambda_{j-1, s}} \right)^* = \frac{\lambda_{j, s}}{\lambda_{j-1}}$$

for all $j = 1, \dots, m$, we conclude that

$$\begin{aligned}
& \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j_k=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k,s}} \right)^{\frac{1}{\lambda_{k,s}}} \\
&= \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j_k=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1,s} \left(\frac{p}{\lambda_{k-1,s}} \right)^*} \right)^{\frac{1}{\lambda_{k-1,s} \left(\frac{p}{\lambda_{k-1,s}} \right)^*}} \\
&= \left\| \left(\left(\sum_{\widehat{j_k=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1,s}} \right)_{j_k=1}^n \right\|_{\left(\frac{p}{\lambda_{k-1,s}} \right)^*}^{\frac{1}{\lambda_{k-1,s}}} \\
&= \left(\sup_{y \in B_{\ell_p^n}^{\frac{p}{\lambda_{k-1,s}}}} \sum_{j_k=1}^n |y_{j_k}| \left(\sum_{\widehat{j_k=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1,s}} \right)^{\frac{1}{\lambda_{k-1,s}}} \\
&= \left(\sup_{x \in B_{\ell_p^n}^{\frac{p}{\lambda_{k-1,s}}}} \sum_{j_k=1}^n |x_{j_k}|^{\lambda_{k-1,s}} \left(\sum_{\widehat{j_k=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k-1,s}} \right)^{\frac{1}{\lambda_{k-1,s}}} \\
&= \sup_{x \in B_{\ell_p^n}^{\frac{p}{\lambda_{k-1,s}}}} \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j_k=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s} \lambda_{k-1,s}} \right)^{\frac{1}{\lambda_{k-1,s}}} \\
&\leq C_m \|T\|.
\end{aligned}$$

where the last inequality holds by (20).

- $i \neq k$.

It is clear that $\lambda_{k-1,s} < \lambda_{k,s}$ for all $1 \leq k \leq m$. Since $\lambda_{k,s} < s$ for all $1 \leq k \leq m$ (see (19)) we get

$$\lambda_{k-1,s} < \lambda_{k,s} < s \text{ for all } 1 \leq k \leq m.$$

Denoting, for $i = 1, \dots, m$,

$$S_i = \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s}}$$

we get

$$\begin{aligned}
& \sum_{j_i=1}^n \left(\sum_{\widehat{j_i=1}}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k,s}} = \sum_{j_i=1}^n S_i^{\lambda_{k,s}} = \sum_{j_i=1}^n S_i^{\lambda_{k,s}-s} S_i^s \\
&= \sum_{j_i=1}^n \sum_{\widehat{j_i=1}}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k,s}}} = \sum_{j_k=1}^n \sum_{\widehat{j_k=1}}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k,s}}} \\
&= \sum_{j_k=1}^n \sum_{\widehat{j_k=1}}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^{\frac{s(s-\lambda_{k,s})}{s-\lambda_{k-1,s}}}}{S_i^{s-\lambda_{k,s}}} |T(e_{j_1}, \dots, e_{j_m})|^{\frac{s(\lambda_{k,s}-\lambda_{k-1,s})}{s-\lambda_{k-1,s}}}.
\end{aligned}$$

Therefore, using Hölder's inequality twice we obtain

$$\begin{aligned}
 (21) \quad & \sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k,s}} \\
 & \leq \sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}} \right)^{\frac{s-\lambda_{k,s}}{s-\lambda_{k-1,s}}} \left(\sum_{\widehat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{\lambda_{k,s}-\lambda_{k-1,s}}{s-\lambda_{k-1,s}}} \\
 & \leq \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}} \right)^{\frac{\lambda_{k,s}}{\lambda_{k-1,s}}} \right)^{\frac{\lambda_{k-1,s}}{\lambda_{k,s}} \cdot \frac{s-\lambda_{k,s}}{s-\lambda_{k-1,s}}} \times \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k,s}} \right)^{\frac{1}{\lambda_{k,s}} \cdot \frac{(\lambda_{k,s}-\lambda_{k-1,s})s}{s-\lambda_{k-1,s}}} .
 \end{aligned}$$

We know from the case $i = k$ that

$$(22) \quad \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k,s}} \right)^{\frac{1}{\lambda_{k,s}} \cdot \frac{(\lambda_{k,s}-\lambda_{k-1,s})s}{s-\lambda_{k-1,s}}} \leq (C_m \|T\|)^{\frac{(\lambda_{k,s}-\lambda_{k-1,s})s}{s-\lambda_{k-1,s}}} .$$

Now we investigate the first factor in (21). From Hölder's inequality and (20) it follows that

$$\begin{aligned}
 (23) \quad & \left(\sum_{j_k=1}^n \left(\sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}} \right)^{\frac{\lambda_{k,s}}{\lambda_{k-1,s}}} \right)^{\frac{\lambda_{k-1,s}}{\lambda_{k,s}}} = \left\| \left(\sum_{\widehat{j}_k} \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}} \right)_{j_k=1}^n \right\|_{\left(\frac{p}{\lambda_{k-1,s}}\right)^*} \\
 & = \sup_{y \in B_{\ell_p^n}^{\frac{p}{\lambda_{k-1,s}}}} \sum_{j_k=1}^n |y_{j_k}| \sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}} = \sup_{x \in B_{\ell_p^n}} \sum_{j_k=1}^n \sum_{\widehat{j}_k=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}} |x_{j_k}|^{\lambda_{k-1,s}} \\
 & = \sup_{x \in B_{\ell_p^n}} \sum_{j_i=1}^n \sum_{\widehat{j}_i=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^{s-\lambda_{k-1,s}}}{S_i^{s-\lambda_{k-1,s}}} |T(e_{j_1}, \dots, e_{j_m})|^{\lambda_{k-1,s}} |x_{j_k}|^{\lambda_{k-1,s}} \\
 & \leq \sup_{x \in B_{\ell_p^n}} \sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n \frac{|T(e_{j_1}, \dots, e_{j_m})|^s}{S_i^s} \right)^{\frac{s-\lambda_{k-1,s}}{s}} \left(\sum_{\widehat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s} \lambda_{k-1,s}} \\
 & = \sup_{x \in B_{\ell_p^n}} \sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s |x_{j_k}|^s \right)^{\frac{1}{s} \lambda_{k-1,s}} \leq (C_m \|T\|)^{\lambda_{k-1,s}} .
 \end{aligned}$$

Replacing (22) and (23) in (21) we conclude that

$$\begin{aligned}
 \sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{k,s}} & \leq (C_m \|T\|)^{\lambda_{k-1,s} \frac{s-\lambda_{k,s}}{s-\lambda_{k-1,s}}} (C_m \|T\|)^{\frac{(\lambda_{k,s}-\lambda_{k-1,s})s}{s-\lambda_{k-1,s}}} \\
 & = (C_m \|T\|)^{\lambda_{k,s}}
 \end{aligned}$$

and finally the proof of (16) is done for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1} \right)$.

Now the proof uses a different argument from those from [4], since a new interpolation procedure is now needed. From (19) we know that $\lambda_{m,s} < s$ for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$. Therefore, using the Minkowski inequality as in [1], it is possible to obtain from (16) that, for all fixed $i \in \{1, \dots, m\}$,

$$(24) \quad C_{m,p,(s,\dots,s,\lambda_{m,s},s,\dots,s)}^{\mathbb{K}} \leq \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}$$

for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$ with $\lambda_{m,s}$ in the i -th position. Finally, from Lemma 2.2 we know that $(q_1^{-1}, \dots, q_m^{-1})$ belongs to the convex hull of

$$\{(\lambda_{m,s}^{-1}, s^{-1}, \dots, s^{-1}), \dots, (s^{-1}, \dots, s^{-1}, \lambda_{m,s}^{-1})\}$$

for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$ with certain constants $\theta_{1,s}, \dots, \theta_{m,s}$ and thus, from the interpolative technique from [1], we get

$$\begin{aligned} C_{m,p,\mathbf{q}}^{\mathbb{K}} &\leq \left(C_{m,p,(\lambda_{m,s},s,\dots,s)}^{\mathbb{K}}\right)^{\theta_{1,s}} \cdots \left(C_{m,p,(s,\dots,s,\lambda_{m,s})}^{\mathbb{K}}\right)^{\theta_{m,s}} \\ &\leq \left(\prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}\right)^{\theta_{1,s}+\dots+\theta_{m,s}} \\ &= \prod_{j=2}^m A_{\frac{2j-2}{j}}^{-1}. \end{aligned}$$

Now we prove (ii), which is simpler.

Define

$$s_{\mathbf{q}} = \max q_i$$

and, for $s \in \left[\frac{2mp}{mp+p-2m}, 2\right]$,

$$\lambda_{0,s} = \frac{2s}{ms + s + 2 - 2m}$$

and

$$(25) \quad \lambda_{m,s} = \frac{2ps}{mps + ps + 2p - 2mp - 2ms}.$$

Since $\left[\frac{2mp}{mp+p-2m}, 2\right] \subseteq \left[\frac{2mp-2p}{mp-2m}, 2\right]$ we have $\frac{p}{p-m} \leq \lambda_{m,s} \leq 2$ (see (8)) and since $\left[\frac{2mp}{mp+p-2m}, 2\right] \subseteq \left[\frac{2m-2}{m}, 2\right]$ from (11) we know $1 \leq \lambda_{0,s} \leq 2$.

We prove the case of real scalars. For complex scalars the proof is analogous, and we can replace $\sqrt{2}$ by $\frac{2}{\sqrt{\pi}}$ and \mathbb{R} by \mathbb{C} . Since

$$\frac{m-1}{s} + \frac{1}{\lambda_{0,s}} = \frac{m+1}{2},$$

from the generalized Bohnenblust–Hille inequality (see [1]) we know that there is a constant $C_m \geq 1$ such that for all m -linear forms $T : \ell_{\infty}^n \times \dots \times \ell_{\infty}^n \rightarrow \mathbb{K}$ we have, for all $i = 1, \dots, m$,

$$(26) \quad \left(\sum_{j_i=1}^n \left(\sum_{\hat{j}_i=1}^n |T(e_{j_1}, \dots, e_{j_m})|^s \right)^{\frac{1}{s} \lambda_{0,s}} \right)^{\frac{1}{\lambda_{0,s}}} \leq C_m \|T\|.$$

Since

$$\frac{2m}{m+1} \leq \frac{2mp}{mp+p-2m} \leq s \leq 2,$$

we know that

$$(27) \quad \lambda_{0,s} \leq s \leq 2.$$

To verify the first inequality in (27) we just need to repeat the argument used to prove (13), now supposing $s \geq \frac{2m}{m+1}$.

The multiple exponent

$$(\lambda_{0,s}, s, s, \dots, s)$$

can be obtained by interpolating the multiple exponents $(1, 2, \dots, 2)$ and $(\frac{2m}{m+1}, \dots, \frac{2m}{m+1})$ with, respectively,

$$\begin{aligned} \theta_1 &= 2 \left(\frac{1}{\lambda_{0,s}} - \frac{1}{s} \right), \\ \theta_2 &= m \left(\frac{2}{s} - 1 \right), \end{aligned}$$

in the sense of [1].

The exponent $(\frac{2m}{m+1}, \dots, \frac{2m}{m+1})$ is the classical exponent of the Bohnenblust–Hille inequality and the estimate of the constant associated to $(1, 2, \dots, 2)$ is $(\sqrt{2})^{m-1}$ (see, for instance, [4], although this result is very well-known).

Therefore, the optimal constant associated to the multiple exponent

$$(\lambda_{0,s}, s, s, \dots, s)$$

is less than or equal (for real scalars) to

$$\left((\sqrt{2})^{m-1} \right)^{2 \left(\frac{1}{\lambda_{0,s}} - \frac{1}{s} \right)} (B_{\mathbb{R},m}^{\text{mult}})^{m \left(\frac{2}{s} - 1 \right)}$$

i.e.,

$$C_m \leq \left(\sqrt{2} \right)^{2(m-1) \left(\frac{m+1}{2} - \frac{m}{s} \right)} (B_{\mathbb{R},m}^{\text{mult}})^{m \left(\frac{2}{s} - 1 \right)}.$$

More precisely, (26) is valid with C_m as above. For complex scalars we can use the Khinchine inequality for Steinhaus variables and replace $\sqrt{2}$ by $\frac{2}{\sqrt{\pi}}$ as in [20]. Therefore, analogously to the previous case (see also [4, Theorem 1.1]), it is possible to prove that

$$(28) \quad C_{m,p,(\lambda_{m,s}, s, \dots, s)}^{\mathbb{R}} \leq \left(\sqrt{2} \right)^{2(m-1) \left(\frac{m+1}{2} - \frac{m}{s} \right)} (B_{\mathbb{R},m}^{\text{mult}})^{m \left(\frac{2}{s} - 1 \right)}$$

for all $s \in \left[\frac{2mp}{mp+p-2m}, 2 \right]$.

Since $s \geq \frac{2mp}{mp+p-2m}$ we have $\lambda_{m,s} \leq s$ (in fact, we just need to imitate the argument used to prove (9), now supposing $s \geq \frac{2mp}{mp+p-2m}$) and so from (28), using the Minkowski inequality as in [1], it is possible to obtain, for all fixed $j \in \{1, \dots, m\}$,

$$(29) \quad C_{m,p,(s, \dots, s, \lambda_{m,s}, s, \dots, s)}^{\mathbb{R}} \leq \left(\sqrt{2} \right)^{2(m-1) \left(\frac{m+1}{2} - \frac{m}{s} \right)} (B_{\mathbb{R},m}^{\text{mult}})^{m \left(\frac{2}{s} - 1 \right)}$$

for all $s \in \left[\frac{2mp}{mp+p-2m}, 2 \right]$ with $\lambda_{m,s}$ in the j -th position. Therefore, given $\epsilon > 0$ (sufficiently small), consider

$$s_{\mathbf{q}+\epsilon} := s_{\mathbf{q}} + \epsilon = \max q_i + \epsilon,$$

and since $s_{\mathbf{q}+\epsilon} > \frac{2mp}{mp+p-2m}$ (because $s_{\mathbf{q}} = \max q_i \geq \frac{2mp}{mp+p-2m}$) we have (29) for $s = s_{\mathbf{q}+\epsilon}$. Finally, from Lemma 2.2 we know that $(q_1^{-1}, \dots, q_m^{-1})$ belongs to the convex hull of

$$\left\{ \left(\lambda_{m,s_{\mathbf{q}+\epsilon}}^{-1}, s_{\mathbf{q}+\epsilon}^{-1}, \dots, s_{\mathbf{q}+\epsilon}^{-1} \right), \dots, \left(s_{\mathbf{q}+\epsilon}^{-1}, \dots, s_{\mathbf{q}+\epsilon}^{-1}, \lambda_{m,s_{\mathbf{q}+\epsilon}}^{-1} \right) \right\}$$

with certain constants $\theta_{1,s_{\mathbf{q}+\epsilon}}, \dots, \theta_{m,s_{\mathbf{q}+\epsilon}}$ and thus, from the interpolative technique from [1], we get

$$\begin{aligned} & C_{m,p,\mathbf{q}}^{\mathbb{R}} \\ & \leq \left(C_{m,p,(\lambda_{m,s_{\mathbf{q}+\epsilon}}, s_{\mathbf{q}+\epsilon}, \dots, s_{\mathbf{q}+\epsilon})}^{\mathbb{R}} \right)^{\theta_{1,s_{\mathbf{q}+\epsilon}}} \dots \left(C_{m,p,(s_{\mathbf{q}+\epsilon}, \dots, s_{\mathbf{q}+\epsilon}, \lambda_{m,s_{\mathbf{q}+\epsilon}})}^{\mathbb{R}} \right)^{\theta_{m,s_{\mathbf{q}+\epsilon}}} \\ & \leq \left((\sqrt{2})^{2(m-1)\left(\frac{m+1}{2} - \frac{m}{s_{\mathbf{q}+\epsilon}}\right)} (B_{\mathbb{R},m}^{\text{mult}})^{m\left(\frac{2}{s_{\mathbf{q}+\epsilon}} - 1\right)} \right)^{\theta_{1,s_{\mathbf{q}+\epsilon}} + \dots + \theta_{m,s_{\mathbf{q}+\epsilon}}} \\ & = (\sqrt{2})^{2(m-1)\left(\frac{m+1}{2} - \frac{m}{s_{\mathbf{q}+\epsilon}}\right)} (B_{\mathbb{R},m}^{\text{mult}})^{m\left(\frac{2}{s_{\mathbf{q}+\epsilon}} - 1\right)} \end{aligned}$$

for all $\epsilon > 0$ sufficiently small. By making $\epsilon \rightarrow 0$ we get the result. \square

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